Inverse Hyperbolic Tangent Model of Fatigue-Crack Growth

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A fatigue-crack-propagation model was specifically developed to account for the effects of stress ratio and closely represent the sigmoidal shape of the crack-growth-rate curve. This derived functional relation was fitted to large sets of fatigue-crack-propagation data for 2024-T3, 7075-T6, and 7075-T7351 aluminum alloys, and for Ti-6Al-4V alloy taken from the literature. A statistical comparison was made between the functional relation and some commonly used fatigue-crack-propagation models as applied to these data sets. Improved representation was obtained in all cases by using the inverse hyperbolic-tangent-function model.

Nomenclature

a	= crack length, in.
$\mathrm{d}a/\mathrm{d}N$	= crack-growth rate, in./cycle
C,n'	= Paris regression coefficients
C_1,C_2	= regression coefficients
g	= specimen geometric scaling function
K_c	= terminal stress-intensity factor, ksi-(in.)$^{1/2}$
K_{eff}	= effective stress-intensity factor, ksi-(in.)$^{1/2}$
$K_{\max}$	= maximum stress-intensity factor, ksi-(in.)$^{1/2}$
K_0	= threshold stress-intensity factor, ksi-(in.)$^{1/2}$
M'	= slope of scaling function
m	= Walker coefficient
N	= number of loading cycles
n	= number of data points
R	= stress ratio
r^2	= proportion of variation explained by regression
S	= nominal stress, ksi
SSD	= standard error of estimate
U	= function relation accounting for effect of R
W	= specimen width, in.
Y	= dependent variable
Φ	= scaling function
Φ_I	= intercept of scaling function

Introduction

THE determination of the safelife of an aerospace structure exposed to fluctuating loading conditions should be based on a detailed knowledge of the entire continuum of damage mechanisms. Thus, a quantitative knowledge of both the fatigue and fatigue-crack-propagation processes is necessary for safe and effective structural design. Unfortunately, the study of fatigue-crack-growth phenomena has been hampered by the lack of an accurate physical model of crack-tip damage accumulation. Investigators have overcome this difficulty through the use of the various empirical formulas that have been developed for fatigue-crack propagation. It is the purpose of this paper to describe the formulation of such a phenomenological fatigue-crack-propagation model.

Laboratory studies have been conducted to obtain fatigue-crack-propagation data for various materials. Extensive tests have been performed utilizing center-cracked, compact-tension, and surface-flaw specimens. Data have been generated on high-strength steel, aluminum, and titanium alloys under both constant- and variable-amplitude loading conditions. (The present investigation is concerned only with constant-amplitude results.) Fatigue-crack-propagation data, recorded in the form of (a_i,N_i) are not directly useful for design purposes since a variety of stress levels, stress ratios, initial crack conditions, and environmental conditions are encountered. To make use of the above data, a fatigue-crack-growth rate model based on these data is used to account for the effects of stress ratio on crack-growth rate.

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Empirical Representation of Crack Propagation

In general, the relationship between crack size and number of applied cycles can be represented as a crack-growth curve drawn through the raw data points. Although data presented in this form indicate general trends, it has been found convenient to model crack-damage behavior as a rate process and formulate a dependent variable based on the slope of this growth curve. The instantaneous rate of change of crack length, or an approximation to it

$$Y = \mathrm{d}a/\mathrm{d}N \cong \Delta a/\Delta N \tag{1}$$

has been chosen as the dependent variable for the formulation of fatigue-crack-propagation models.

Results obtained from the theory of linear elasticity have suggested that the damage occurring at the crack tip might be represented as a function of a stress-intensity factor which, in general form, may be written as

$$K_{\max} = S(a\pi)^{1/2} g(a,W) \tag{2}$$

As a result, the independent variable is usually considered as some function of K and stress ratio.

If the slope of the crack-growth curve is calculated at these same points, then the locus of points $[(\mathrm{d}a/\mathrm{d}N),\ K]$ can be plotted. These variables are generally plotted on log-log scales to obtain a crack-growth-rate curve. Examination of this curve suggests several factors of importance that will have to be accounted for in the formulation of a crack-growth model.

In most materials, there is an upper limit to the crack severity and associated critical stress-intensity factor which a material can sustain. At this critical value, the crack will propagate unstably. For the rate diagram of $\mathrm{d}a/\mathrm{d}N$ vs $K_{\max}$, the K_c value is the terminal (or upper) limit on the abscissa and thus, the rate of crack growth becomes very large as $K_{\max}$ approaches K_c. At the other extreme, a minimum crack-growth rate of zero is anticipated at a zero value of $K_{\max}$. However, this assumption appears to be conservative since evidently there exists a threshold below which no fatigue-crack propagation occurs. Thus, the doubly logarithmic plot of $\mathrm{d}a/\mathrm{d}N$ vs $K_{\max}$ reveals a curve having sigmoidal shape.

Within the general curve shape previously described, systematic variations in the data point locations are observed. When data from tests conducted at several different stress ratios are present, the plot of crack-growth rate vs stress-intensity factor will be layered into distinct bands above the locus of points having zero stress ratio; the positive ratio bands lying above and the negative ratio bands lying below. Layering of data points also occurs as a result of variation in such parameters as test frequency, environment, and specimen grain direction. It is desirable to predict the characteristic effects of each parameter; thus, researchers have formulated mathematical models accounting for at least some of these parametric effects.

Assuming the variables $K_{\max}$, R, and $\mathrm{d}a/\mathrm{d}N$, the general form for a fatigue-crack-propagation model has been established as

$$\mathrm{d}a/\mathrm{d}N = f(K_{\max}, R) \tag{3}$$

The following discussion describes efforts to obtain a useful functional form for $f(K_{\max}, R)$.

Structure of the Modeling Problem

The basic concepts discussed in the previous section suggest that the modeling procedure can be thought of as consisting of three distinct steps: 1) Calculation of dependent variable values—how can the crack-growth rates be best calculated from the discrete (a_i, N_i) data points? 2) Formulation of an independent variable—what combination of R and K can best be used to formulate an independent variable that will consolidate the crack-growth-rate data? 3) Formulation of an analytical model—what functional form containing the dependent and independent variables should be chosen to best approximate the sigmoidal character of the crack-growth-rate curve? The last two parts of the modeling problem were addressed in this study.

Particular effort was devoted to obtaining an expression in which compensations for the effect of stress ratio were uncoupled from the factors influencing general curve shape. Such a feature permits a greater flexibility in the analysis of fatigue-crack-propagation data.

Formulation of the Independent Variable

It was previously suggested that the independent variable be some function of $K_{\max}$ and R. As a general form for the independent variable, assume

$$K_{\mathrm{eff}}(K_{\max}, R) = U(R) K_{\max} \tag{4}$$

A number of different forms for $U(R)$ have been proposed. The simplest of these is $U(R) = 1.0$. In this way, it is asserted that no stress ratio effects are present; then

$$K_{\mathrm{eff}} = K_{\max} \tag{5}$$

This relation is appropriate if no variation in stress ratio is contained in the data, or if the material is insensitive to changes in stress ratio. The stress-intensity range also may be used as an independent variable. Letting $U(R) = (1-R)$, the expression

$$K_{\mathrm{eff}} = (1-R) K_{\max} \tag{6}$$

results. This relation has been widely used in the past. Walker[1] proposed that the independent variable should represent a combination of maximum stress intensity and stress-intensity range. Letting $U(R) = (1-R)^m$, K_{eff} has the form

$$K_{\mathrm{eff}} = (1-R)^m K_{\max} \tag{7}$$

The selection of a form for the independent variable should be based on physical insights as well as on the statistical performance displayed in fitting experimental data. It has been observed that most materials exhibit stress-ratio dependent behavior, thus it is reasonable to assume that the choice of $U(R) = 1.0$ would seldom be satisfactory. It would not, on the other hand, be reasonable to assume that $U(R) = (1-R)$, i.e., that the behavior is governed only by the stress-intensity range. The Walker formulation, a combination of these dependencies, is thus a more physically justifiable selection than either alone. It was selected for use as an independent variable in the fatigue-crack-propagation model.

Linear models, of necessity, neglect initial and terminal behavior. The general form for this linear model is

$$\log \frac{\mathrm{d}a}{\mathrm{d}N} = \log C + n' (\log K_{\mathrm{eff}}) \tag{8}$$

Best known of these models is the law of Paris[5]

$$\log \frac{\mathrm{d}a}{\mathrm{d}N} = \log C + n' \log [(1-R) K_{\max}] \tag{9}$$

This equation is commonly fitted to the data to yield the Paris regression coefficients C and n'. The Paris model incorporates a term to account for the effect of stress ratio. It has been used extensively in the literature.

Other linear models are possible, and several have been proposed. These relations, which must be considered elaborations of the Paris model, include those due to Elber,[6] Walker,[1] and Roberts and Erdogan.[7] The primary differences in these expressions lie in the choice of the independent variable.

Modifications of the linear Paris model have been made to create a nonlinearity at the terminal end of the curve. To approximate the sigmoidal character of the rate curve, and to better account for the effects of stress ratio, Forman[8] proposed the relation

$$\frac{\mathrm{d}a}{\mathrm{d}N} = \frac{C[(1-R)K]^{n'}}{(1-R)(K_c - K)} \tag{10}$$

Forman's equation contains a singular term in the denominator to model the terminal region of crack growth. As K approaches the critical stress intensity, the denominator goes to zero. Manipulation of the Forman equation leads to

$$\begin{aligned} \frac{\mathrm{d}a}{\mathrm{d}N} &= \frac{(1-R)^{n'}}{(1-R)} \left[\frac{CK^{n'}}{(K_c - K)} \right] \\ &= \left[\frac{C(1-R)^{n'-1} K^{n'}}{K_c - K} \right] \end{aligned} \tag{11}$$

The term $(1-R)^{n'}$ is clearly similar to the Walker formulation for the independent variable and as such helps to account for the effect of stress ratio. Forman's equation has no provision for modeling the existence of a threshold of crack growth. Variations on the form of the singularity are possible.

The other approach to the modeling of the crack-propagation process is to assume a nonlinear function. Recently, Collipriest[9] suggested a fatigue-crack-propagation law to model the entire rate curve. This nonlinear equation is based on the inverse hyperbolic-tangent function. In Collipriest's equation, the independent variable takes the form of $K_{\mathrm{eff}} = \Delta K$. This nonlinear approach was investigated further because it seemed to provide a realistic method for analysis of fatigue-crack-propagation data. Rather than utilizing Collipriest's equation, it was decided to derive a fatigue-crack-propagation model that would allow the implementation of the most effective of the independent variable formulations described earlier. The goal of this derivation was also to obtain a more compact analytical form for the fatigue-crack-propagation model.

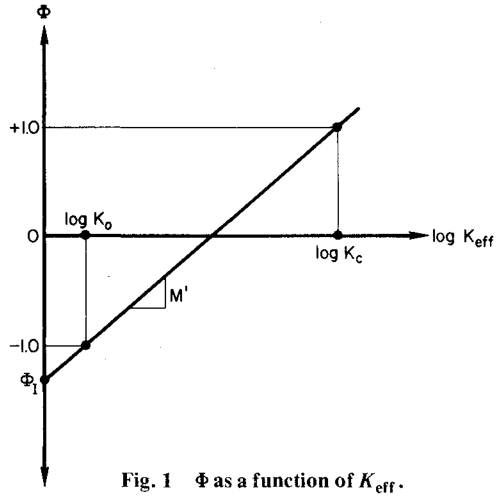

Fig. 1 Φ **as a function of** K_{eff}.

Present Model

The model was based on the inverse hyperbolic tangent suggested by Collipriest. The functional form assumed was

$$\log \frac{da}{dN} = C_1 + C_2 \tanh^{-1}[\Phi(K_{eff})] \tag{12}$$

Examination of the $\tanh^{-1}$ curve suggested the proper form for $\Phi(K_{eff})$. The function $\Phi(K_{eff})$ was chosen to scale values of the effective stress-intensity factor into values in the range of the argument of $\tanh^{-1}$. The $\tanh^{-1}$ function goes to infinity at the values of $\Phi = -1$ and $\Phi = +1$. Clearly, the regions of rapid rate of change of da/dN should correspond to arguments in the neighborhood of ± 1. To establish this correspondence, a function was assumed to scale the K_{eff} values into the interval $\Phi = -1$ to $\Phi = +1$.

The physical initial and final values were assigned to the points ($\log K_0, -1$) and ($\log K_c, +1$) on the $\Phi - \log K$ plane as illustrated in Fig. 1. Assuming a linear scaling function

$$\Phi = M' \log K_{eff} + \Phi_I \tag{13}$$

the slope and the intercept were determined by applying the previous conditions. This yielded a system of simultaneous, linear, algebraic equations,

$$1 = M' (\log K_c) + \Phi_I \tag{14a}$$

$$-1 = M' (\log K_0) + \Phi_I \tag{14b}$$

Solving these for the slope and intercept and substituting into Eq. (14) gives the result

$$\Phi(K_{eff}) = \frac{\log (K_c K_0 / K_{eff}^2)}{\log (K_0 / K_c)} \tag{15}$$

and the basic form of the fatigue-crack-propagation model becomes

$$\log \frac{da}{dN} = C_1 + C_2 \tanh^{-1}\left[\frac{\log (K_c K_0 / K_{eff})^2}{\log (K_0 / K_c)} \right] \tag{16}$$

Completion of the fatigue-crack-propagation model required that a form for K_{eff} be specified. Since the Walker formulation for K_{eff} was chosen, the complete fatigue-crack-propagation model is

$$\log \frac{da}{dN} = C_1 + C_2 \tanh^{-1}\left[\frac{\log [K_c K_0 / (K_{max}(1-R)^m)^2]}{\log (K_0 / K_c)} \right] \tag{17}$$

Application of the Inverse Hyperbolic-Tangent Model

To examine the effectiveness of Eq. (17) in analyzing fatigue-crack-propagation data, the model was applied to data sets compiled on several different materials by means of a computer program. Starting with encoded (a_i, N_i) data, this program fitted the inverse hyperbolic-tangent model to [$K_{max\ i}$, $da/dN|_i$] values in the following steps: 1) Crack-propagation rates were evaluated by a five-point divided-difference scheme. 2) Maximum stress-intensity-factor values were calculated using the appropriate fracture mechanics formula for the given specimen type. 3) Values of the argument

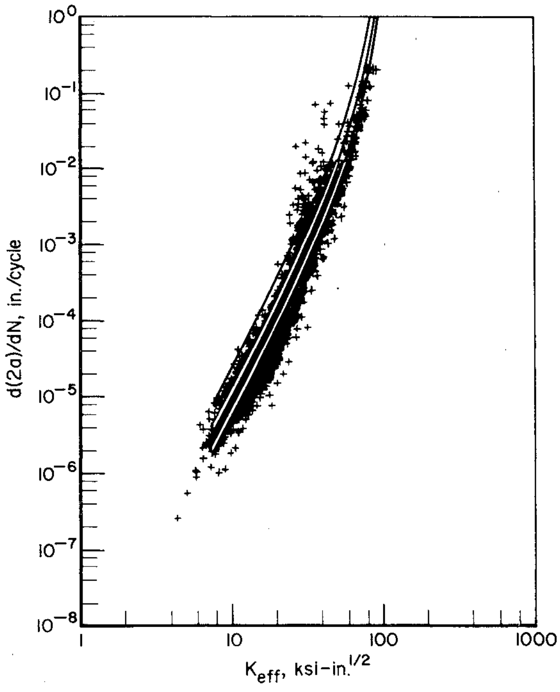

Fig. 2 Fatigue-crack propagation rate curve for 2024-T3.

Table 1 Results of regression analysis and coefficients for Eq. (17)

Material	n	C_1	C_2	m	K_0, $MN/m^{3/2}$ (ksi-in.$^{1/2}$)	K_c, $MN/m^{3/2}$ (ksi-in.$^{1/2}$)	r^2	SSD	$S_{y.x}$	Reference numbers
2024-T3	3407	-4.489	3.465	0.420	2.20 (2.00)	142.74 (130.00)	0.923	273.00	0.255	13,14,16 18,19,20, 21,22,23
7075-T6	746	-4.207	2.241	0.320	3.29 (3.00)	85.64 (78.00)	0.912	47.28	0.252	12,13,14, 15,15
7075-T7351	1,082	-4.043	2.574	0.350	4.36 (4.00)	109.90 (100.00)	0.952	33.81	0.177	17
Ti-6A1-4V	782	-4.046	2.825	0.580	4.39 (4.00)	274.50 (250.00)	0.982	36.14	0.215	24,25

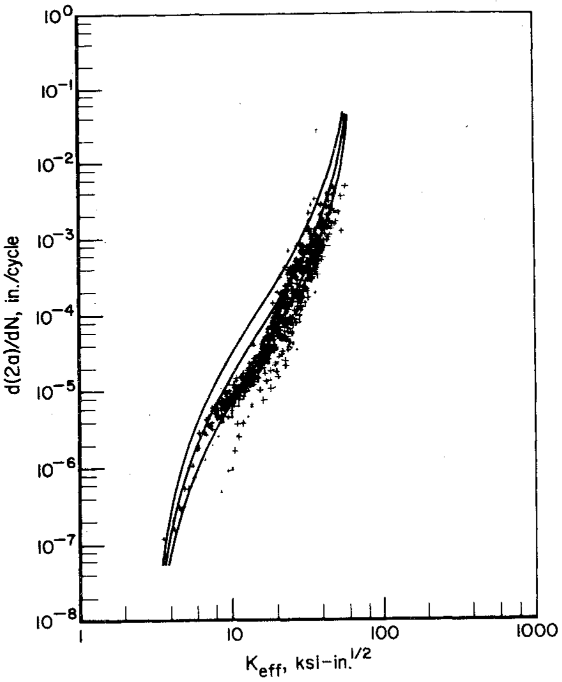

Fig. 3 Fatigue-crack-propagation rate curve for 7075-T6.

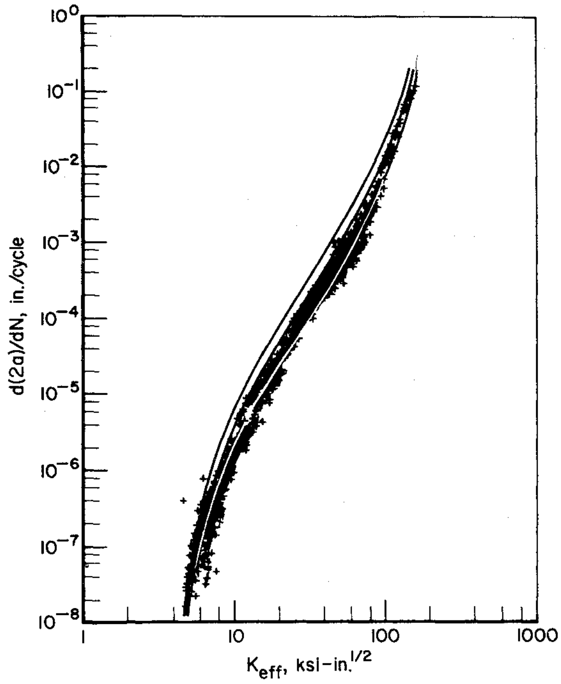

Fig. 5 Fatigue-crack-propagation rate curve for Ti-6A1-4V.

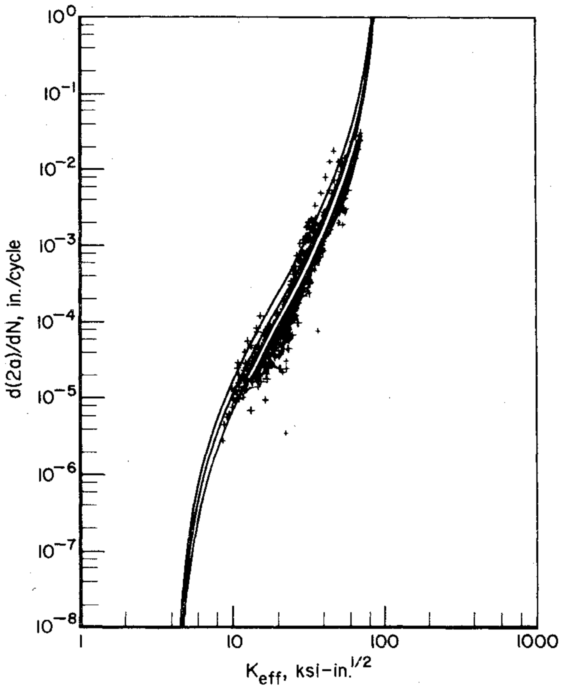

Fig. 4 Fatigue-crack-propagation rate curve for 7075-T7351.

$\Phi(K_{eff})$ were calculated from the K_{max} results. 4) Equation (17) was fitted to the $[\Phi(K_{eff}), (da/dN)|_i]$ values by least-squares regression. The coefficient m was optimized by minimizing the SSD value through at iterative procedure. 5) Statistical parameters, including SSD, r^2, and S, were generated. Tolerance limits were computed.

Analysis of the data necessitated a selection for the values of K_0 and K_c. An excellent summary of threshold values is presented in the paper by Donahue, et al.[10] Data are included in this source on a large number of materials. Values of K_c can be found in such publications as the Damage Tolerant Handbook.[11] These sources yielded average values for the K_0 and K_c limits on the crack-growth-rate curve.

Nominal values for K_0 were selected from the paper by Donahue, et al.[10] Nominal values for K_c were established by inspection of the K_{max} values for the data sets. Data on four materials were analyzed; 2024-T3, 7075-T6, and 7075-T7351 aluminum, and Ti-6A1-4V alloy. The composition of these four data sets are listed in Table 1.

Detailed results of the regression analysis performed are presented in Table 1. Number of data points, regression coefficients computed with English units, K_0 and K_c values, and statistical parameters are presented. Figures 2-5 show the consolidated fatigue-crack-propagation data, the fitted curve and tolerance limits. Good characterizations of the data were obtained in most cases. Particularly satisfactory results were achieved for the titanium alloy (Fig. 2). The sigmoidal charac-

Table 2 Comparison of fatigue-crack-propagation models for complete data sets

	Values of r^2		
Material	$\log \frac{da}{dN} = C+n' \log K(1-R)$	$C+n' \log(1-R)K - \log(1-R)(K_c-K)$	$C_1+C_2 \tanh^{-1}\Phi$
2024-T3	0.829	0.877	0.923
7075-T6	0.669	0.875	0.912
7075-T7351	0.880	0.926	0.952
Ti-6A1-4V	0.939	0.970	0.982

ter of this crack-growth-rate curve is clearly displayed by these data.

A comparison between three methods of fatigue-crack-propagation analysis was made. The five data sets were regressed in three different ways; with the inverse hyperbolic—tangent model, with the Paris model [Eq. (9)], and with the Forman model [Eq. (10)]. The results of the comparison are presented in Table 2. From this table it is observed that the inverse hyperbolic-tangent model provided improved representation of the data, when compared with the other two methods.

Conclusion

The inverse hyperbolic tangent model presents an effective means for describing fatigue-crack-propagation data. It is believed that several areas in investigation should be pursued in refining the method. Among these are: 1) establishing a physical basis for the choice of the $\tanh^{-1}$ model to describe the propagation phenomena; 2) establishing a rational procedure for selecting K_c values; and 3) integrating Eq. (17) to yield predictions of specimen/structure life, and compare methods of analysis on this basis.

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